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TECHNICAL NOTE 3646

A THEORY FOR THE ELASTIC DEFLECTIONS OF PLATES INTEGRALLY STIFFENED ON ONE SIDE

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SUMMARY

An elastic-deflection theory is presented for anisotropic plates which exhibit coupling between bending and stretching. In particular, the theory provides a basis for analyzing plates integrally stiffened on one side. The effect of coupling is responsible for the presence of added terms in the potential-energy expression, the small- and large-deflection equations of equilibrium, and the boundary conditions. Example calculations show that in the case of a simply supported square plate loaded in compression and having equal flexural stiffnesses in the principal directions, coupling may significantly lower the in-plane compressive load at which deflections grow rapidly.

INTRODUCTION

Plates integrally stiffened on one side are shown in references 1, 2, and 3 to be capable of carrying higher loads than conventional unstiffened plates. Stiffening on one side entails, however, a characteristic coupling between in-plane strains and lateral deflections of the plate which may lower the compressive load that the plate elements of a structure could carry if coupling were not present. Also, if such plate elements form portions of a wing surface, the aerodynamic surface might be adversely affected by the coupled deflections.

The presence of coupling in integrally stiffened plates, such as those shown in figure 1, is recognized in references 4 and 5, but the effect of coupling on the load-carrying characteristics of a plate has not been previously investigated. A system of equations applicable to integrally stiffened plates is given in the present paper. The potential-energy expression, the differential equation of equilibrium, and the natural boundary conditions are presented to provide two approaches to analysis. An example is also included which illustrates the deflections of a simply supported square plate acted upon by compressive loads which are less than the uncoupled critical load.

Much of the material presented in the present paper was originally included in a thesis submitted to the Virginia Polytechnic Institute in partial fulfillment of the requirements for the master of science degree in Applied Mechanics in May 1955.

SYMBOLS

a	length of plate
b	width of plate
$C_{11}, C_{12}, C_{21}, C_{22}, C_K$	coupling constants defined in equations (1), in.
D	flexural stiffness of isotropic plate, in-lb
D_{xy}	twisting stiffness relative to x- and y-directions, in-lb
D_1, D_2	flexural stiffnesses in x- and y-directions, respectively, in-lb
E_1, E_2	extensional stiffnesses in x- and y-directions, respectively, lb/in.
F	force function defined by equation (9)
G_K	shearing stiffness in xy-plane, lb/in.
H	overall thickness of plate plus integral stiffeners, in.
k	edge loading parameter, $-\bar{N}_x b^2/\pi^2 D$
M_x, M_y	resultant bending-moment intensities acting on cross sections originally perpendicular to x- and y-axes, respectively, lb
M_{xy}	resultant twisting-moment intensity acting on cross sections originally perpendicular to x- and y-axes, lb
$\bar{M}_x, \bar{M}_y, \bar{M}_{xy}$	boundary values of resultant bending and twisting moments, lb
N_x, N_y	resultant normal-force intensities acting in planes I and II of cross sections originally perpendicular to x- and y-axes, respectively, lb/in.

N_{xy}	resultant shear-force intensity acting in plane III of cross sections originally perpendicular to x- and y-axes, lb/in.
$\bar{N}_x$, $\bar{N}_{xy}$	boundary values of resultant normal and shear forces, lb/in.
n	positive odd integers
Q_x, Q_y	resultant transverse shear-force intensities acting on cross sections parallel to yz-plane and xz-plane, respectively, lb/in.
$\bar{Q}_x$, $\bar{Q}_y$	boundary values of resultant transverse shear-force intensities acting in z-direction on plane originally perpendicular to x- and y-axes, respectively, lb/in.
q	lateral-loading intensity, psi
u, v, w	components of displacement in x-, y-, and z-directions, respectively, in.
V	total potential energy of system, in-lb
X_n	components in x-direction for small-deflection solution to differential equation of equilibrium
x, y, z	orthogonal coordinate system; z measured normal to plane of plate, and x and y measured parallel to axes of principal stiffness
γ_{xy}	shear strain in plane III with respect to x- and y-directions
ϵ_x, ϵ_y	normal strain in plane I in x-direction and in plane II in y-direction, respectively
μ_x, μ_y	Poisson's ratios associated with bending in x- and y-directions, respectively, and defined by equations (1)
μ_1, μ_2	Poisson's ratios associated with extensions in x- and y-directions, respectively, and defined by equations (1)

BASIC EQUATIONS

Coupling, as used herein, refers to the interaction between strains in the plane of the plate and lateral deflections of the plate. The stress distribution through the thickness of the skin and integral stiffener on one side of a plate stiffened on one side will not be symmetric because the construction is not symmetric. This lack of symmetry will introduce bending and twisting moments which produce curvature when direct stresses are applied in the plane of the plate.

A complete elastic-deflection theory for plates exhibiting coupling can be composed of the following sets of equations:

1. Force-distortion equations

2. Strain-displacement equations

3. Differential equations of equilibrium

4. Displacement and stress-resultant boundary conditions

The force-distortion equations (equations relating stress resultants to strains and curvatures) are given in reference 4 for integrally stiffened plates. These equations are discussed subsequently to provide completeness in the present paper. The equations which relate strains and curvatures to displacements are given in reference 6 (p. 342) and are presented herein. The equilibrium equations in terms of the resultant moments and in-plane forces are the same as those in ordinary plate theory and are given in reference 6 (pp. 299-300). The boundary conditions will be specified by the particular problem being considered.

For some applications, the potential-energy expression is used in preference to solving the differential equation of equilibrium. The potential-energy expression is therefore derived in the appendix and presented in the section entitled "Potential-Energy Expression."

Force-Distortion Equations

References 4 and 5 include the effects of coupling in equations relating forces, moments, strains, and curvatures, and present methods for calculating the associated elastic constants and coupling terms for integrally stiffened plates. A general form of the force-distortion equations given in reference 4 is

$$\left.\begin{aligned}
M_x &= -D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) + C_{11}N_x + C_{12}N_y \\
M_y &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2}\right) + C_{21}N_x + C_{22}N_y \\
M_{xy} &= D_{xy} \frac{\partial^2 w}{\partial x\,\partial y} + C_K N_{xy} \\
\epsilon_x &= C_{11} \frac{\partial^2 w}{\partial x^2} + C_{21} \frac{\partial^2 w}{\partial y^2} + \frac{N_x}{E_1} - \frac{\mu_2}{E_2} N_y \\
\epsilon_y &= C_{12} \frac{\partial^2 w}{\partial x^2} + C_{22} \frac{\partial^2 w}{\partial y^2} - \frac{\mu_1}{E_1} N_x + \frac{N_y}{E_2} \\
\gamma_{xy} &= -2C_K \frac{\partial^2 w}{\partial x\,\partial y} + \frac{N_{xy}}{G_K}
\end{aligned}\right\} \tag{1}$$

The terms containing the C's are the added terms due to coupling and the C's are the coupling constants. The quantities $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ are the average curvatures and $\partial^2 w/\partial x\,\partial y$ is the average twist. The planes in which the forces N_x, N_y, and N_{xy} act do not necessarily coincide with the midplane of the plate. These forces are applied in arbitrarily located planes I, II, and III, respectively, and the strains ϵ_x, ϵ_y, and γ_{xy} are measured in the corresponding planes. Figure 2 illustrates the forces and moments acting on an element of an equivalent, uniform thickness, anisotropic plate. The material properties of this idealized element may be considered to vary unsymmetrically about the midplane to permit coupling to exist even when the resultant forces lie in the midplane. In reference 4 the locations of the planes in which the forces act are left arbitrary for the sake of generality. The force-distortion equations (see eqs. (1)) apply only to cases in which the axes of principal stiffness coincide with the directions of the resultant forces.

Some of the coupling terms in the force-distortion equations may be eliminated by locating the planes in which the resultant forces act so as to cause a moment, and thus a curvature, counter to that given by the coupling effect. That is, plane I may be chosen so that either C_{11} or C_{21} is zero; plane II may be chosen so that either C_{22} or C_{12} is zero; plane III may be chosen so that C_K is zero. In general, three of the coupling constants may be eliminated. For the special case of a flat homogeneous plate or a plate symmetrically stiffened on both sides, all coupling is, of course, eliminated by locating planes I, II, and III in the midplane of the plate.

The general form of the force-distortion equations as given in equations (1) is used in the derivation of the potential-energy expression, the equilibrium equations, and the natural boundary conditions.

Strain-Displacement Equations

The strain-displacement equations applicable to this theory which considers the effects of coupling are the same as those given in reference 6 (p. 342) for ordinary plates. They are

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned}\right\} \tag{2}$$

Equilibrium Equations

The equilibrium equations for a plate which deforms according to equations (1) will be the same as those for any plate theory when they are expressed in terms of the resultant in-plane forces and moments.

Small-deflection equation of equilibrium.- The small-deflection equation of equilibrium is given in reference 6 (p. 300) as

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{3}$$

which may be written as

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\left(q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\right) \tag{4}$$

where

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} \tag{5}$$

and

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} \tag{6}$$

In small-deflection analysis the forces in the plane of the plate are assumed to be unaffected by lateral deflections and will be known functions. Therefore, the equations of equilibrium for the forces in the plane of the plate (see ref. 6, p. 299) remain satisfied during deflection and are, accordingly, neglected.

By substituting the expressions in equations (1) for the moments into equation (3), the effect of coupling on the equilibrium equation is exhibited.

Large-deflection equations of equilibrium.- For a large-deflection analysis where the in-plane forces may no longer be considered to be independent of lateral deflections, all the equilibrium equations must be considered simultaneously with the strain-displacement and force-distortion equations. Combining these three sets of equations in a manner similar to that given in reference 6 (pp. 342-343) results in the following two equations:

$$\frac{1}{E_2}\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_K} - 2\frac{\mu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_1}\frac{\partial^4 F}{\partial y^4} + C_{12}\frac{\partial^4 w}{\partial x^4} + \left(C_{11} + C_{22} + 2C_K\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} +$$
$$C_{21}\frac{\partial^4 w}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{7}$$

$$D_1\frac{\partial^4 w}{\partial x^4} + 2\left(\mu_y D_1 + D_{xy}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2\frac{\partial^4 w}{\partial y^4} - C_{12}\frac{\partial^4 F}{\partial x^4} -$$
$$\left(C_{11} + C_{22} + 2C_K\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} - C_{21}\frac{\partial^4 F}{\partial y^4} = q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} \tag{8}$$

where

$$\left.\begin{aligned} N_x &= \frac{\partial^2 F}{\partial y^2} \\ N_y &= \frac{\partial^2 F}{\partial x^2} \\ N_{xy} &= -\frac{\partial^2 F}{\partial x\,\partial y} \end{aligned}\right\} \tag{9}$$

Equations (7) and (8), together with the boundary conditions, determine w and F. The loading intensities may then be determined from equations (9).

Boundary Conditions

The boundary conditions to be imposed along any edge of a plate will fall into two classes: namely, conditions on displacements or conditions on forces. These conditions will always occur in pairs such that either a condition on a displacement or a condition on the associated force will be prescribed, but never will both be prescribed.

For the case of a rectangular plate with dimensions such as those shown in figure 3, one of each of the following pairs of quantities may be prescribed along the indicated edges. (Note that these quantities do not apply for elastically supported edges.)

Displacement		Force	
Along edges $x = 0$ and $x = a$:			
w	or	$Q_x - \dfrac{\partial M_{xy}}{\partial y}$	(10)
$\dfrac{\partial w}{\partial x}$	or	M_x	(11)
Along edges $y = 0$ and $y = b$:			
w	or	$Q_y - \dfrac{\partial M_{xy}}{\partial x}$	(12)
$\dfrac{\partial w}{\partial y}$	or	M_y	(13)

For cases in which the force is the prescribed component, coupling will affect the boundary conditions. This is illustrated in the section entitled "Illustrative Examples."

Potential-Energy Expression

The derivation of the general potential-energy expression for a plate that deforms according to the force-distortion relations given in equations (1) is presented in the appendix. The derivation of the expression is applicable only to cases in which the reactions do no work. The results, therefore, may be used only for combinations of free, simply supported, or clamped-edge conditions. A more general expression may be obtained by considering the energy of elastic restraint distributed along the edges of the plate. A method for including this generalization in the potential-energy expression is presented in reference 7.

The expression for the total potential energy of a rectangular plate with edges $x = 0$ and a and $y = 0$ and b is derived in the appendix as

$$
\begin{aligned}
V = \frac{1}{2} \int_0^a \int_0^b \Bigg[& D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\mu_y \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2D_{xy}\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 + \\
& N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - 2qw\Bigg] dx\, dy - \\
& \frac{1}{2} \int_0^a \int_0^b \Bigg[2\left(C_{11}N_x + C_{12}N_y\right)\frac{\partial^2 w}{\partial x^2} + 2\left(C_{21}N_x + C_{22}N_y\right)\frac{\partial^2 w}{\partial y^2} + \\
& 4C_K N_{xy} \frac{\partial^2 w}{\partial x\, \partial y}\Bigg] dx\, dy - \frac{1}{2} \int_0^a \int_0^b \Bigg[\frac{{N_x}^2}{E_1} - \\
& \left(\frac{\mu_1}{E_1} + \frac{\mu_2}{E_2}\right)N_x N_y + \frac{{N_y}^2}{E_2} + \frac{{N_{xy}}^2}{G_K}\Bigg] dx\, dy + \int_0^b \left|\bar{M}_x \frac{\partial w}{\partial x} - \bar{M}_{xy} \frac{\partial w}{\partial y} - \bar{Q}_x w\right|_0^a dy + \\
& \int_0^a \left|\bar{M}_y \frac{\partial w}{\partial y} - \bar{M}_{xy} \frac{\partial w}{\partial x} - \bar{Q}_y w\right|_0^b dx
\end{aligned}
\tag{14}
$$

The first integral and the last two integrals on the right-hand side of equation (14) are those which would result from the small-deflection analysis of an uncoupled anisotropic plate. The second integral of equation (14) contains the additional terms due to coupling, and the third integral gives the energy of extensional straining in the planes in which the forces N_x, N_y, and N_{xy} act. For small-deflection analysis, the variations of the third integral will vanish because the forces N_x, N_y, and N_{xy} will be prescribed. The second integral, which represents the energy associated with coupling, will remain, however, in small-deflection analysis. For large-deflection analysis, relationships must be derived between displacements and in-plane forces so that the third integral may be evaluated.

DISCUSSION OF APPLICATIONS

There will be some classes of problems for which coupling will have no effect, whereas for others, the effect will be significant. Consider the following cases:

1. The effects of coupling on the stability of plates acted upon by forces in the plane of the plate

2. The effects of coupling on plates acted upon by lateral loads

3. The effects of coupling on the distribution of forces in the plane of the plate

The effect of coupling is most apparent in problems which involve stability considerations in the uncoupled case because coupling can lower the load which the plate can carry without excessive deflection. Coupling causes lateral deflection as soon as the forces are applied in the plane of the plate, in a manner quite similar to the deflection caused by eccentric loading. When lateral deflection occurs upon application of the in-plane forces, the problem may no longer be considered as one of stability in the usual sense, but must be treated as a deflection problem. At some value of the applied forces, small-deflection analysis will indicate a rapid growth of deflection for a small increase in the applied forces, and this level may be defined as the onset of instability. The boundary conditions are shown in the section entitled "Illustrative Examples" to be very important in that, under certain conditions, the effects of coupling are eliminated.

The effect of coupling on the deflection of a plate under lateral load is less significant than for the previously discussed case. For lateral loading, coupling only alters the already-present membrane stresses which are usually neglected anyway. It is therefore likely that the membrane forces may be neglected in the presence of coupling also.

In the stability analysis of conventional plates, the stress distribution due to the boundary forces is found by assuming that the plate remains flat during loading and that the redistribution of the forces due to a small deflection is negligible. In the presence of coupling there will be some additional redistribution of the forces during deflection. It is likely that the total redistribution may be neglected here also because of two considerations. First, small-deflection problems of practical interest will not involve severe changes in the curvature of the plate from one point to another. (Changes in curvature cause changes in the forces. See eqs. (1).) Second, the proportions of the forseeable, practical types of integrally stiffened plates will be such that the coupling constants will be small enough so as not to affect the stress distribution seriously. If cases arise which require consideration of the redistribution of the stresses, the analysis will probably be more difficult than the conventional large-deflection analysis because of the added complexity due to coupling.

For many of the practical problems in which coupling must be considered, the loading will be simple; that is, uniform compression, uniform shearing, and in-plane bending. As discussed previously, the redistribution of the forces will be negligible. A small-deflection analysis may be made by using the potential-energy expression, equation (14), or by solving the differential equation of equilibrium, equation (3), subject to the appropriate boundary conditions. By substituting the moments from equations (1) into equation (3), the equilibrium equation becomes becomes

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2\left(\mu_y D_1 + D_{xy}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - C_{11} \frac{\partial^2 N_x}{\partial x^2} - C_{12} \frac{\partial^2 N_y}{\partial x^2} +$$

$$2C_K \frac{\partial^2 N_{xy}}{\partial x\, \partial y} - C_{21} \frac{\partial^2 N_x}{\partial y^2} - C_{22} \frac{\partial^2 N_y}{\partial y^2} = q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x\, \partial y} \quad (15)$$

The terms due to coupling will drop out of the equilibrium equation for many of the practical problems. The equilibrium equation will become that for the conventional anisotropic plate; only the coupling terms which may occur in the boundary conditions (eqs. (10) to (13)) will remain to impose the effects of coupling on the problem.

ILLUSTRATIVE EXAMPLES

For any case in which the coupling terms drop out of the equilibrium equation and all the edges of the plate are clamped, there will be no effect of coupling because the displacement boundary conditions will be specified. (See eqs. (10) to (13).) That is, none of the coupling terms will appear in the boundary conditions and there will, therefore, be no effect of coupling.

In the case of a simply supported rectangular plate acted upon by a uniformly distributed load $\bar{N}_{xy}$ there will be no effect of coupling. That is, the displacements will be specified in equations (10) and (12) which contain no effects of coupling and the forces will be specified in equations (11) and (13) which impose no effects of coupling because N_x and N_y are zero. Thus, this uniform-shear case may be treated as though coupling were not present.

Consider now the case of a simply supported rectangular plate uniformly loaded in compression as shown in figure 3. The coupling terms are eliminated from the equilibrium equation, equation (15), as previously discussed. Displacements will again be specified in equations (10) and (12) which involve no coupling; forces will be specified in equations (11) and (13) which will involve coupling. That is, equations (11) and (13) become, after substitution from equations (1),

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0,a} = \frac{C_{11}N_x}{D_1}$$

and

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = \frac{C_{21}N_x}{D_2}$$

In order to simplify this example, consider $C_{21} = 0$, $D_1 = D_2 = \mu_y D_1 + D_{xy} = D$, and $a/b = 1$. Mathematically, the problem is then reduced to solving the equilibrium equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}$$

subject to the boundary conditions

$$w\Big|_{x=0,a} = w\Big|_{y=0,b} = 0$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = 0$$

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0,a} = \frac{C_{11}N_x}{D}$$

This solution may be found to be

$$w = \sum_{n=1,3,5,\ldots}^{\infty} X_n \sin\frac{n\pi y}{a}$$

where

$$X_n = \frac{2C_{11}k}{\pi n^3 a_n b_n}\left(\frac{\sin n\pi b_n \sinh \frac{n\pi}{a} a_n x \cos \frac{n\pi}{a} b_n x + \sinh n\pi a_n \cosh \frac{n\pi}{a} a_n x \sin \frac{n\pi}{a} b_n x}{\cos n\pi b_n + \cosh n\pi a_n} - \sinh \frac{n\pi}{a} a_n x \sin \frac{n\pi}{a} b_n x\right)$$

and

$$a_n = \frac{1}{2}\sqrt{4 - \frac{k}{n^2}}$$

$$b_n = \frac{1}{2n}\sqrt{k}$$

$$k = -\frac{\bar{N}_x b^2}{\pi^2 D}$$

Figure 4 shows a plot of the deflections of the midpoint of a square plate against values of the load parameter k. The value of the coupling term chosen, $C_{11} = 0.050H$, is a feasible value, yet the calculations indicate significant deflections at loads well below $k = 4.00$, which corresponds to the buckling value for the uncoupled plate. For example, the plate of reference 8 which had longitudinal ribbing 0.212 inch high on 0.063-inch-thick skin had a calculated value of C_{11} of about 0.002H. An equivalent-weight plate with 0.212-inch-high 45° ribbing (see fig. 1(c)) on 0.063-inch-thick skin, however, would have a value of C_{11} of approximately -0.086H.

It may be noted that the effect of the coupling term in this example is analagous to moments of intensity $-C_{11}N_x/D$ acting along the edges $x = 0$ and $x = a$ of an uncoupled plate. The deflections produced by these moments would form a mode which is similar to the uncoupled natural buckling mode. This similarity may intensify the effects of coupling in this particular example. A reduced effect may be anticipated for larger plate aspect ratios.

CONCLUDING REMARKS

The elastic-deflection theory presented herein forms a basis for the analysis of anisotropic plates exhibiting coupling between bending and stretching. In particular, the theory provides a basis for analyzing plates integrally stiffened on one side. The effects of coupling are included in the potential-energy expression, the small- and large-deflection equilibrium equations, and the boundary conditions.

The most significant effect of coupling is that it can, depending upon the boundary conditions, lower the load which the plate could carry without excessive deflections if coupling were not present. Coupling causes bending upon application of load, and a deflection analysis is required for this type of problem rather than a stability analysis.

Example calculations show that coupling causes significant deflection of a compressed, simply supported square plate having equal flexural stiffnesses in the two orthogonal directions at loads well below the critical buckling load for the equivalent, uncoupled plate. The uncoupled natural buckling mode is the same in this example as the mode of deflection caused by the coupling. In cases where this is not so, the results may be quite different and, therefore, deserve further study.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
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APPENDIX

DERIVATION OF POTENTIAL-ENERGY EXPRESSION

The derivation of the potential-energy expression is applicable to rectangular plates whose edges are either free, simply supported, or clamped. The potential-energy expression is first derived for the general case in order to include in the derivation the effects of coupling which arise from strains in the plane of the plates. The terms which are not included for small-deflection analysis are then indicated.

The potential-energy expression for the case of small deflections may be obtained, in a manner similar to that presented in reference 7, as

$$V = \frac{1}{2}\int_0^a\int_0^b\left[-M_x\frac{\partial^2 w}{\partial x^2} - M_y\frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2qw\right]dx\,dy + \int_0^b\left|\bar{M}_x\frac{\partial w}{\partial x} - \bar{M}_{xy}\frac{\partial w}{\partial y} - \bar{Q}_x w\right|_0^a dy + \int_0^a\left|\bar{M}_y\frac{\partial w}{\partial y} - \bar{M}_{xy}\frac{\partial w}{\partial x} - \bar{Q}_y w\right|_0^b dx \tag{A1}$$

The potential energy may be made general by adding the energy of stretching in the plane of the plate. This expression is given in reference 6 (p. 303) as

$$-\frac{1}{2}\int_0^a\int_0^b\left(N_x\epsilon_x + N_y\epsilon_y + N_{xy}\gamma_{xy}\right)dx\,dy \tag{A2}$$

By substituting the moments from equations (1) into equation (A1) and the strains from equations (1) into expression (A2) and by adding the two resulting expressions, the potential energy is obtained in the following form which will probably be the most useful for analysis:

$$V = \frac{1}{2}\int_0^a\int_0^b\left[D_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\mu_y\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2D_{xy}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2qw\right]dx\,dy - \frac{1}{2}\int_0^a\int_0^b\left[2\left(C_{11}N_x + C_{12}N_y\right)\frac{\partial^2 w}{\partial x^2} + 2\left(C_{21}N_x + C_{22}N_y\right)\frac{\partial^2 w}{\partial y^2} + 4C_K N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\right]dx\,dy - \frac{1}{2}\int_0^a\int_0^b\left[\frac{N_x^2}{E_1} - \left(\frac{\mu_1}{E_1} + \frac{\mu_2}{E_2}\right)N_xN_y + \frac{N_y^2}{E_2} + \frac{N_{xy}^2}{G_K}\right]dx\,dy + \int_0^b\left|\bar{M}_x\frac{\partial w}{\partial x} - \bar{M}_{xy}\frac{\partial w}{\partial y} - \bar{Q}_x w\right|_0^a dy + \int_0^a\left|\bar{M}_y\frac{\partial w}{\partial y} - \bar{M}_{xy}\frac{\partial w}{\partial x} - \bar{Q}_y w\right|_0^b dx \tag{A3}$$

For small-deflection analysis, only the third term on the right-hand side of equation (A3) may be omitted. The forces N_x, N_y, and N_{xy} will be prescribed in small-deflection analysis, and that term containing only the forces will accordingly drop out when the variation of the potential energy is taken.

The in-plane equilibrium equation (see ref. 6, p. 299) has been used in deriving equation (A3) from equation (A1).

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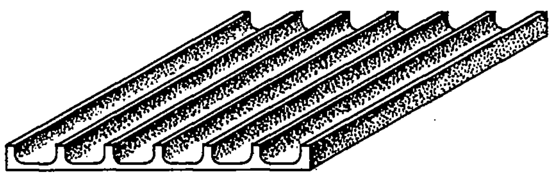
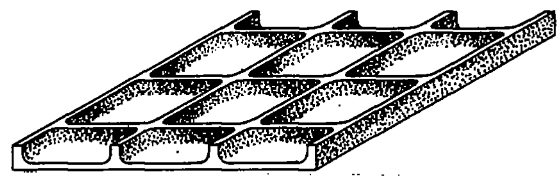
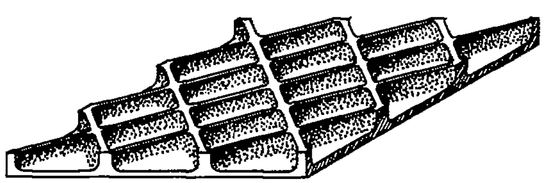


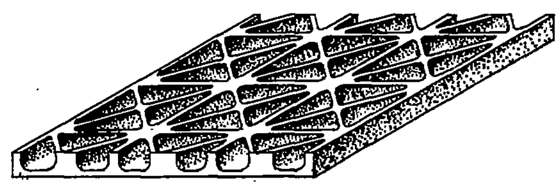
(a) Longitudinal or transverse.



(b) Longitudinal and transverse.



(c) Skewed.



(d) Skewed plus longitudinal and transverse.

Figure 1.– Examples of integrally stiffened plates.

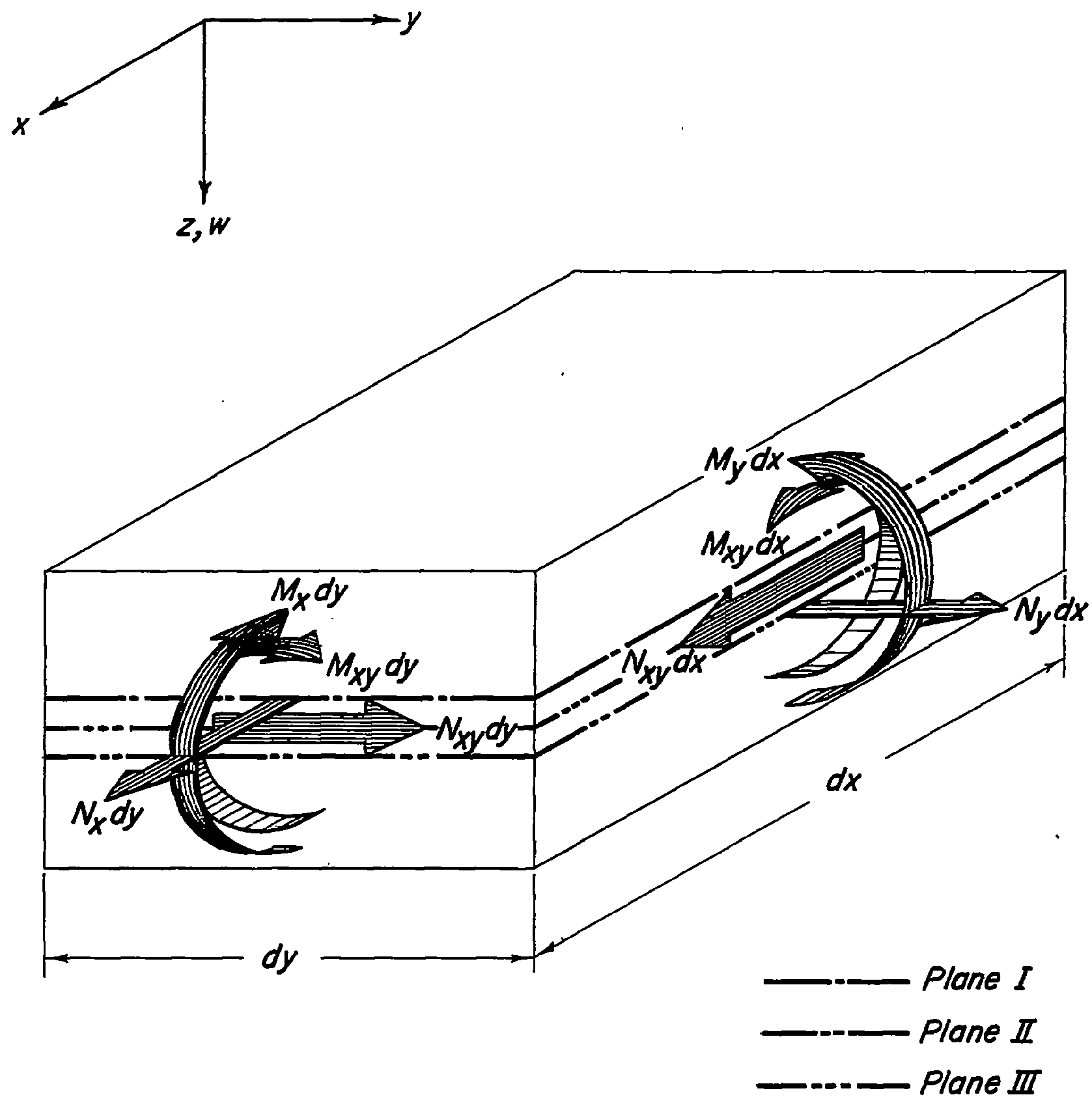


Figure 2.- Forces and moments acting on element of an equivalent, uniform thickness, anisotropic plate.

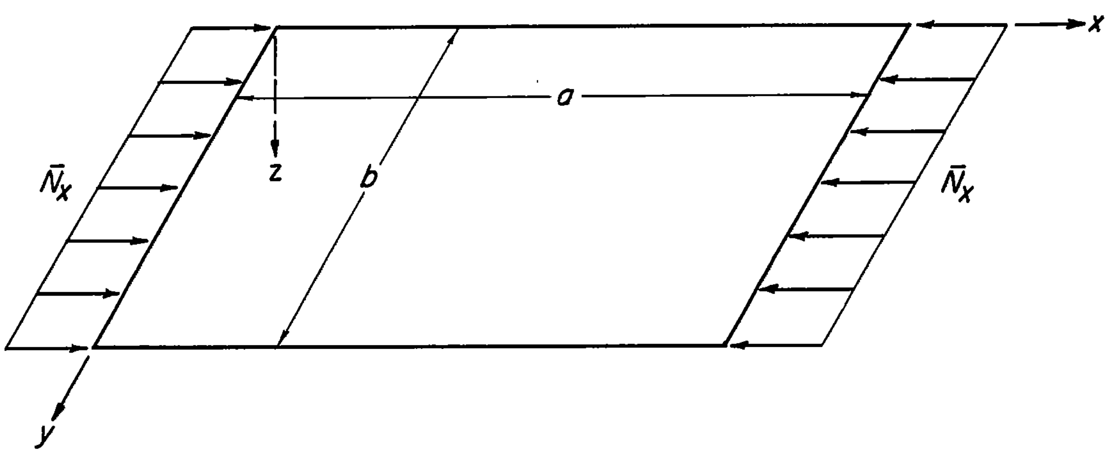


Figure 3.- Simply supported plate considered in analysis.

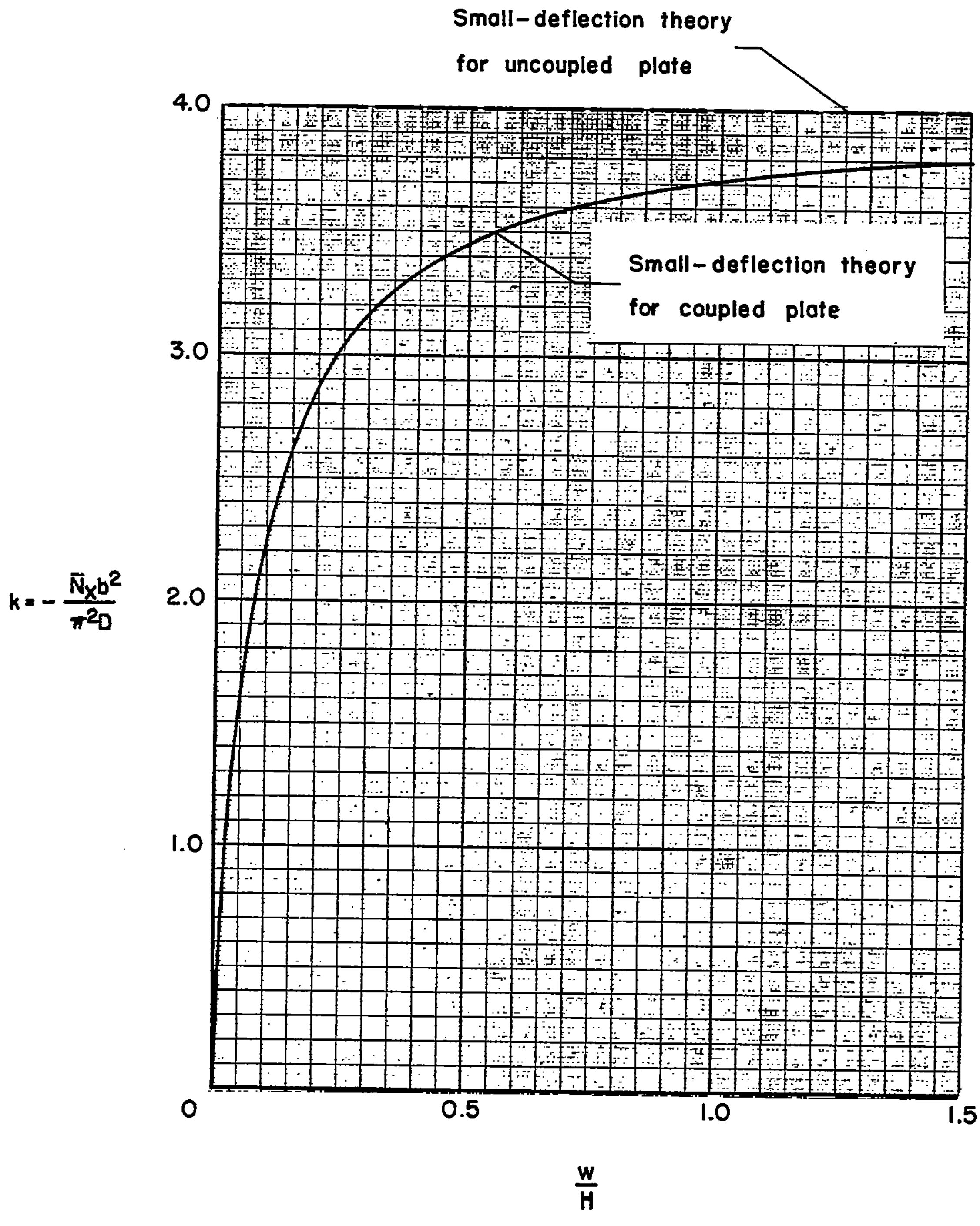


Figure 4.- Variation with load of midpoint deflection of simply supported square plate. $D_1 = D_2 = \mu_y D_1 + D_{xy} = D$; $a/b = 1$; $C_{11} = 0.050H$; $C_{21} = 0$.